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AN OPTIMUM SHADOW SHIELD

The Nuclear/Chemical Pulse Reaction Propulsion Project

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AN OPTIMUM SHADOW SHIELD

The method of Lagrange multipliers is used to design an optimum shadow shield for an isotropic point source. An exponential absorption without multiple scattering is assumed valid within the shield. If the shape of the outer surface is chosen, the inner surface is determined for a given attenuation. Once all the constraints are established (position, attenuation, solid-angle) the derived shape has a minimum mass.

INTRODUCTION

Situations often occur in shield design where it is necessary to minimize the shield weight subject to certain requirements. Among these are a specified total attenuation within a given solid angle, a shield position at a given distance from the source, and a specified shape of the outer surface. Other more restrictive conditions may exist, but these three adequately define the situation for such a large class of problems that it is believed a formalized and easily used solution would be of value to shield designers. In this paper a method will be developed for designing an optimum shadow shield in terms of these constraints for an isotropic point source or its equivalent. The method of Lagrange multipliers will be used in connection with the Euler-Lagrange equation of variational calculus (see Refs. 1 and 2). Since this method follows closely the techniques set forth in detail in the references, the mathematical preliminaries have been omitted.

The concept of shadow shielding is valid so long as radiation is not scattered from the surroundings into the solid angle of interest. This is approximately the case in air, particularly so if secondary radiations are negligible or unimportant. Kimel (Ref. 3) has shown that a variational shield calculation can predict shielding to 20% of experimental values, even when scattering is neglected. His calculations apply, however, to a special case of a shield that gives a certain dose rate at a point detector for a one dimensional linear source. A recent paper by Klohr (Ref. 4) has also indicated some rather interesting uses for shadow shielding in space. In the present report, all scattering will be neglected including multiple scattering within the shield; but it will be shown that the correction for build-up, as in the case of gamma rays, can be included if desired.

Following the usual pedagogic style, the solution for the general case will first be developed, followed by three special cases for different shapes of the outer face of the shield; viz., flat, spherical, and parabolic. Several graphs and figures will also be presented as examples.

FORMULATION OF THE PROBLEM

The shape of a generalized shield will be derived giving a specified intensity of radiation at a flat plane located at a distance D from the source. This is shown schematically in Fig. 1 with the plane normal to the X-axis at D. The shield is uniform in density, but its shape as drawn should be considered quite arbitrary, although it is symmetric about the X-axis. It is the quantity t which will be determined as a profile for an optimum shadow shield subtending the solid half-angle ϕ_o. The stationary value of the mass integral will be found subject to the condition of a specified total radiation intensity beyond the shield.

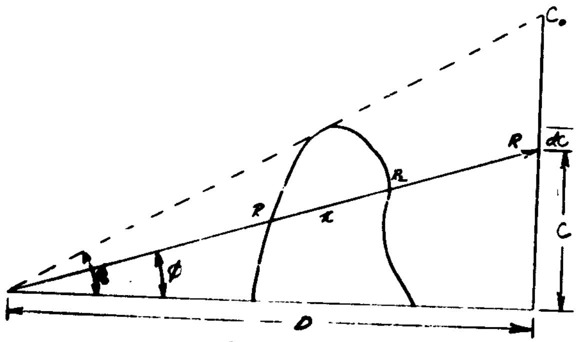

FIGURE 1

Schematic of a Generalized Shadow Shield

The mass of the shield is given by the integral

$$M = \rho V = \rho \int_0^{\phi_o} \int_{R_1}^{R_1+t} 2\pi r^2 \sin\phi \, d\phi \, dr \quad , \tag{1}$$

where

ρ is the material density in g/cm^3.

r is the distance to a mass element in the shield along a radial from O.

ϕ is the angle measured from the X-axis.

R_1 is the distance from the source to the inner face of the shield. (Note that $R_1 + t = R_2$ in Fig. 1.)

and t is the thickness of the shield along a radial, in general t is a function of ϕ.

After performing the integration over r, Eq. 1 becomes

$$M = \int_0^{\phi_o} \frac{2}{3}\pi\rho(t^3 + 3R_1 t^2 + 3R_1^2 t) \sin\phi d\phi \quad . \tag{2}$$

It is the mass integral in this form which will be held stationary in subsequent analysis.

The integral for the intensity of radiation N_D will now be derived in terms of the same variables as in Eq. 2. Before proceeding, however, two definitions will be stated, since they may differ somewhat from other literature:

1. The <u>flux</u> of radiation is meant the number of particles per square centimeter per second.
2. The <u>intensity</u> of radiation is meant the total number of particles per second.

Consider N_o particles emitted isotropically from a point source at O of Fig. 1. On the flat plane at D the intensity is given by the flux times the area; viz.,

$$N_D = \int_0^{C_o}\int_0^{2\pi} \frac{N_o}{4\pi R^2} e^{-\mu t} \cos \phi \; c \; dc \; d\theta \tag{3}$$

As before, all the geometric terms are defined in Fig. 1 except for R, which is the radial distance to the plane of interest, and θ, which is the polar angle on the plane. The area considered is the circle defined by the "shadow" of the shield subtending the solid half-angle ϕ_o.

In Eq. 3 it is assumed that an exponential absorption adequately describes the removal processes within the shield. The cross section μ, then, is the total removal cross section for the type of radiation under consideration. Since gamma rays and neutrons generally obey an exponential law in their penetration of matter, all discussion will be limited to these two types of radiation. It should be understood, however, that the analysis applies to any flux obeying an exponential removal. If build-up is important, as in the case of gamma rays, $e^{-\mu t}$ in Eq. 3 should be replaced by $Be^{-\mu t}$, where B is the build-up factor. Refs. 5 and 6 give B for several different shield materials as a function of incident gamma energy.

By a change of variables with the relations $C = D \tan \phi$, $dc = D \sec^2 \phi \, d\phi$, and $R = D \sec \phi$, Eq. 3 becomes, after performing the integration over θ,

$$N_D = \int_0^{\phi_o} \frac{N_o}{2} e^{-\mu t} \sin \phi d\phi \tag{4}$$

Introducing a Lagrange multiplier λ, the Eqs. 2 and 4 combine to form a modified integrand as in

$$Y = \int_0^{\phi_o} \left\{ \frac{2}{3}\pi \rho \left[t^3 + 3R_1 t^2 + 3R_1^2 t \right] \sin \phi + \frac{\lambda N_o}{2} e^{-\mu t} \sin \phi \right\} d\phi \quad . \tag{5}$$

This equation has the form

$$Y = \int_0^{\phi_o} (I + \lambda J) \, d\phi \quad . \tag{6}$$

For Y to have a stationary value, the integrand $(I + \lambda J)$ must satisfy the Euler-Lagrange equation

$$\frac{\partial (I + \lambda J)}{\partial t} - \frac{d}{d\phi} \frac{\partial (I + \lambda J)}{\partial t_\phi} = 0 \quad . \tag{7}$$

As seen in Eq. 5, $(I + \lambda J)$ is not a function of t_ϕ ; so, the E-L equation reduces to

$$\frac{\partial (I + \lambda J)}{\partial t} = 0 \quad . \tag{8}$$

Differentiating the integrand of Eq. 5 in accordance with Eq. 8 gives

$$2\pi \rho \left[t^2 + 2R_1 t + R_1^2 \right] = \frac{\mu \lambda N_o}{2} e^{-\mu t} \tag{9}$$

or

$$N_o e^{-\mu t} = \frac{4\pi \rho}{\mu \lambda} (t + R_1)^2 = \frac{4\pi \rho}{\mu \lambda} (R_2)^2 \quad . \tag{10}$$

Now substituting Eq. 10 into Eq. 4 gives

$$N_D = \int_0^{\phi_o} \frac{2\pi \rho}{\mu \lambda} R_2^2 \sin \phi d\phi \quad . \tag{11}$$

If the outer surface of the shield R_2 is specified as a function of ϕ such that the integration can be performed, Eq. 11 can be solved for λ. As a final step in deriving the optimum shape, λ is then substituted back into Eq. 10 which is solved for t. It might now be said "quod erat faciendum."

This procedure will now be accomplished for the three special cases cited above.

CASE 1: A FLAT OUTER SURFACE

The problem will now be solved in which the outer shield surface is a flat plane. This is shown schematically in Fig. 2. A more detailed picture is shown in Fig. 4 in the Appendix.

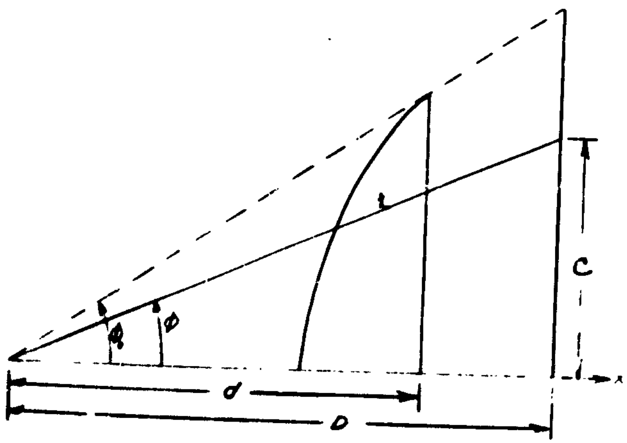

FIGURE 2

... Shield

In this case $R_2 = d \sec \phi$ which gives when substituted into Eq. 11

(12) $$N_D = \int_0^{\phi_o} \frac{2\pi\rho}{\mu\lambda} d^2 \frac{\sin \phi}{\cos^2 \phi} d\phi$$

(13) $$N_D = \frac{2\pi\rho d^2}{\mu\lambda} (\sec \phi_o - 1)$$

or

(14) $$\lambda = \frac{2\pi\rho d^2}{N_D \mu} (\sec \phi_o - 1) \quad .$$

This expression for λ is now substituted into Eq. 10 giving

(15) $$N_o e^{-\mu t} = \frac{4\pi\rho}{\mu\lambda} (R_2)^2 = \frac{4\pi\rho}{\mu\lambda} d^2 \sec^2 \phi$$

(16) $$N_o e^{-\mu t} = 2N_D \frac{\sec^2 \phi}{(\sec \phi_o - 1)}$$

and finally

(17) $$t = \frac{1}{\mu} \ln \frac{N_o}{2N_D} \cos^2 \phi (\sec \phi_o - 1) \quad .$$

This gives the thickness t of a shield which subtends a specified solid half-angle ϕ_o in terms of an "approximate" attenuation factor $N_o/2N_D$. Before discussing this, however, it is interesting to point out that the position of the shield d, originally present, does not enter into the final expression of the shape.* Instead of Eq. 2, the mass integral takes the form

(18) $$M = \int_0^{\phi_o} 2\pi\rho t d^2 \tan \phi \sec \phi \, d\phi \quad .$$

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*In fact if d equals D, the same result is obtained since the constraint in the problem is the intensity of radiation, and not its flux.

This can be easily solved after substitution of t by integration by parts. Doing this is not necessary, though, to see that the <u>absolute</u> minimum mass shield is one in contact with the source; that is, one in which d is a minimum; viz., $d = t$. In practice this is generally not possible, so d then becomes a constraint of the problem along with ϕ_o and the attenuation. Once all three are specified, the derived shield has a minimum mass.

Returning now to the attenuation factor, the shield can attenuate only that fraction of the radiation emitted within the cone of solid half-angle ϕ_o. For an isotropic source this fraction will be denoted by

$$\frac{N_\phi}{N_o} = \frac{1}{4\pi}\int_0^{\phi_o} 2\pi \sin\phi \, d\phi = 1/2(1 - \cos\phi_o) \qquad (19)$$

from which the expression

$$\frac{N_o}{2} = \frac{N_\phi}{1 - \cos\phi_o} \qquad (20)$$

will be substituted into Eq. 17, giving

$$t = \frac{1}{\mu}\ln\left[\frac{N_\phi}{N_D}\cos^2\phi\left(\frac{\sec\phi_o - 1}{1 - \cos\phi_o}\right)\right] \qquad (21)$$

or

$$t = \frac{1}{\mu}\ln\left[\frac{N_\phi}{N_D}\cos^2\phi\sec\phi_o\right] \qquad (22)$$

The quantity N_ϕ/N_D is now the "true" attenuation factor of the shield.

Usually, when referring to attenuation, only the radiation actually incident on the shield is considered. But in some situations, where the intensity through the shield [illegible] important and not its distribution, an

optimum shield is one subtending a solid half-angle less than ϕ_o. This is evident by examining Eq. 22. If the log term is negative, the result is a negative t, which clearly has no physical meaning. It is, therefore, necessary that

$$\left[\frac{N_\phi}{N_D}\cos^2\phi\sec\phi_o\right] \geq 1 \qquad (23)$$

Consider a solid half-angle of $\phi_o = 60^\circ$ in the limiting case of (23) equal to unity. The attenuation factor with $\phi = 60^\circ$ must then be $N_\phi/N_D = 2$. Suppose, however, an attenuation of less than 2 is wanted, say 1.5. In this case (23) equals unity at some angle $\phi < \phi_o$; viz., $\phi \simeq 54.8^\circ$. The shield has a shape similar to that shown in Fig. 3.

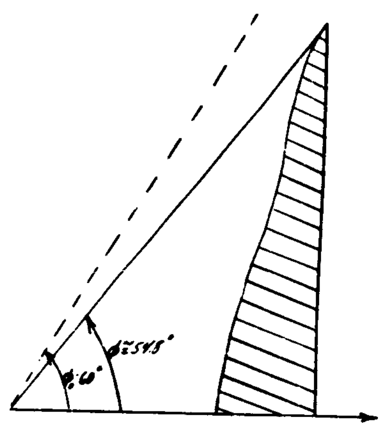

FIGURE 3

Approximate Shape of a Shadow Shield with $\phi_o = 60^\circ$, Attenuation = 1.5

An example of the derived optimum shape is shown in Figs. 5 and 6 and Tables 1 and 2 where several values of μt are plotted and tabulated for a flat faced shield of $\phi_o = 60^\circ$ and 30°, respectively.

CASE 2: A SPHERICAL OUTER SURFACE

In this case the analysis is quite straight forward. In Eq. 11, R_2 is a constant, which gives

$$N_D = \int_0^{\phi_o} \frac{2\pi\rho}{\mu\lambda} R_2^{\,2} \sin\phi d\phi = \frac{2\pi\rho}{\mu\lambda} R_2^{\,2} (1 - \cos\phi_o) \tag{24}$$

or

$$\lambda = \frac{2\pi\rho}{\mu N_D} R_2^{\,2} (1 - \cos\phi_o) \quad . \tag{25}$$

Substitution of Eq. 25 into Eq. 10, and then solving for t is, after cancellation,

$$N_o e^{-\mu t} = 2N_D (1 - \cos\phi_o) \tag{26}$$

$$t = \frac{1}{\mu} \ln \frac{N_o}{2N_D} (1 - \cos\phi_o) \quad . \tag{27}$$

In terms of "true" attenuation as before, Eq. 27, using Eq. 20, reduces to

$$t = \frac{1}{\mu} \ln \frac{N_\phi}{N_D} \tag{28}$$

or

$$N_D = N_\phi e^{-\mu t} \tag{29}$$

Eq. 29 states the usual attenuation law along a radial. The inner shield radius is then, as expected, a constant independent of the angle.

As an example, consider a shadow shield giving an attenuation of $N_\phi/N_D = 5$. The constant thickness μt is then

$$\mu t = \ln N_\phi/N_D \doteq 1.61 \quad .$$

CASE 3: A PARABOLIC OUTER SURFACE

In this case R_2 is given by $R_2 = 2d/1 + \cos\phi$, where d is the distance from the focus of the parabola to its vertex. Eq. 11 is

$$N_D = \int_0^{\phi_o} \frac{8\pi\rho d^2}{\mu\lambda} \frac{\sin\phi}{(1 + \cos\phi)^2} d\phi \quad . \tag{30}$$

A more convenient form is obtained by the transformation $\phi = 2\alpha$. Eq. 30 becomes

$$N_D = \frac{8\pi\rho d^2}{\mu\lambda} \int_0^{\alpha_o} \frac{\sin\alpha}{\cos^3\alpha} d\alpha = \frac{8\pi\rho d^2}{\mu\lambda} \left[\frac{1}{2\cos^2\alpha}\right]_0^{\alpha_o} \tag{31}$$

$$N_D = \frac{4\pi\rho d^2}{\mu\lambda} \left[\frac{1}{\cos^2\alpha_o} - 1\right] \tag{32}$$

or

$$\lambda = \frac{4\pi\rho d^2}{\mu N_D} \left[\frac{1}{\cos^2\alpha_o} - 1\right] \quad . \tag{33}$$

Substituting Eq. 33 into Eq. 10 and solving for t gives

$$N_o e^{-\mu t} = \frac{4N_D}{N_c} \frac{\cos^2\alpha_o}{(\cos\phi + 1)^2 (1 - \cos^2\alpha_o)} \quad , \tag{34}$$

or

$$t = \frac{1}{\mu} \ln \frac{N_o (\cos \phi + 1)^2 (1 - \cos^2 \alpha_o)}{4N_D \cos^2 \alpha_o} \quad . \tag{35}$$

By use of some simple trigonometry, Eq. 35 reduces to

$$t = \frac{1}{\mu} \ln \frac{N_o}{4N_D} (\cos \phi + 1)^2 \left(\frac{1 - \cos \phi_o}{1 + \cos \phi_o}\right) \quad . \tag{36}$$

In terms of the attenuation factor N_ϕ/N_D, defined as before by use of Eq. 20, t becomes

$$t = \frac{1}{\mu} \ln \frac{N_\phi (\cos \phi + 1)^2}{2N_D (\cos \phi_o + 1)} \quad . \tag{37}$$

Several values of μt from Eq. 37 are tabulated in Table 3 for a 60° shadow shield. A comparison of the shapes of each case is shown in Fig. 7.

CONCLUSION

The foregoing relations for μt as a function of attenuation and half angle are examples of an optimum shadow shield which can be derived by the method presented here. In general, any shape shield can be solved for that permits integration of Eq. 11. In this sense then, the method proves quite useful for a very large class of problems, not the least of which, are those encountered in planning the use of nuclear energy in space.

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TABLE 1

Thicknesses of a Flat Faced 60° Shadow Shield for Several Attenuations

ϕ	Radial Thickness - μt					
	N_ϕ/N_D = 2	N_ϕ/N_D = 3	N_ϕ/N_D = 5	N_ϕ/N_D = 25	N_ϕ/N_D = 50	N_ϕ/N_D = 500
0	1.386	1.792	2.303	3.912	4.605	6.908
10	1.356	1.761	2.272	3.881	4.575	6.877
20	1.264	1.668	2.179	3.789	4.482	6.784
30	1.102	1.506	2.018	3.627	4.320	6.623
40	.854	1.291	1.772	3.381	4.074	6.377
50	.5	.908	1.418	3.030	3.721	6.023
60	0	.405	.916	2.526	3.219	5.521

TABLE 2

Thicknesses of a Flat Faced 30° Shadow Shield for Several Attenuations

ϕ	Radial Thickness - μt					
	$N_\phi/N_D = 2$	$N_\phi/N_D = 5$	$N_\phi/N_D = 10$	$N_\phi/N_D = 25$	$N_\phi/N_D = 50$	$N_\phi/N_D = 500$
0	.836	1.754	2.41	3.36	4.05	6.35
10	.798	1.725	2.382	3.29	4.03	6.32
20	.703	1.63	2.278	3.23	3.93	6.23
30	.542	1.468	2.128	3.08	3.87	6.07

TABLE 2

Thicknesses of a Parabolic 60° Shadow Shield for Several Attenuations

φ	Radial Thickness - μt					
	2	5	10	25	50	500
0	1.382	1.392	2.59	3.52	4.20	6.50
10	1.372	1.831	2.53	3.49	4.18	6.48
20	1.365	1.835	2.53	3.42	4.13	6.44
30	1.248	1.757	2.45	3.36	4.06	6.37
40	1.158	1.649	2.34	3.26	3.95	6.25
50	1.03	1.524	2.19	3.11	3.81	6.11
60	.812	1.322	2.02	2.93	3.62	5.92

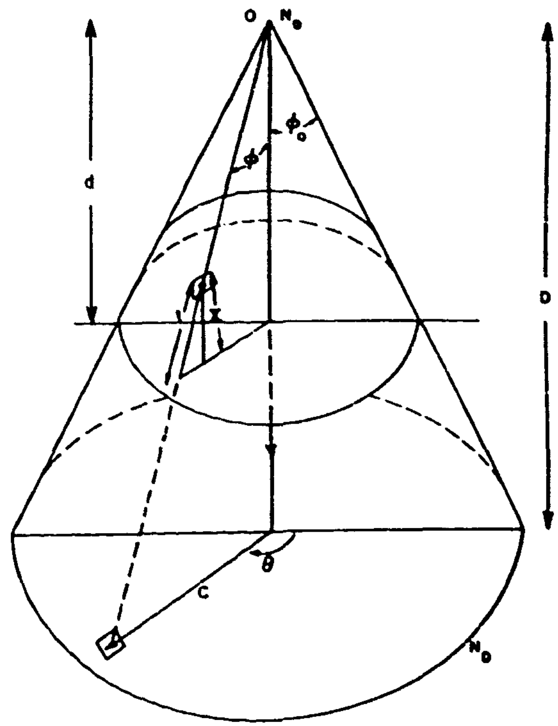

FIGURE 4 THE FLAT FACE SHIELD PROBLEM IN DETAIL

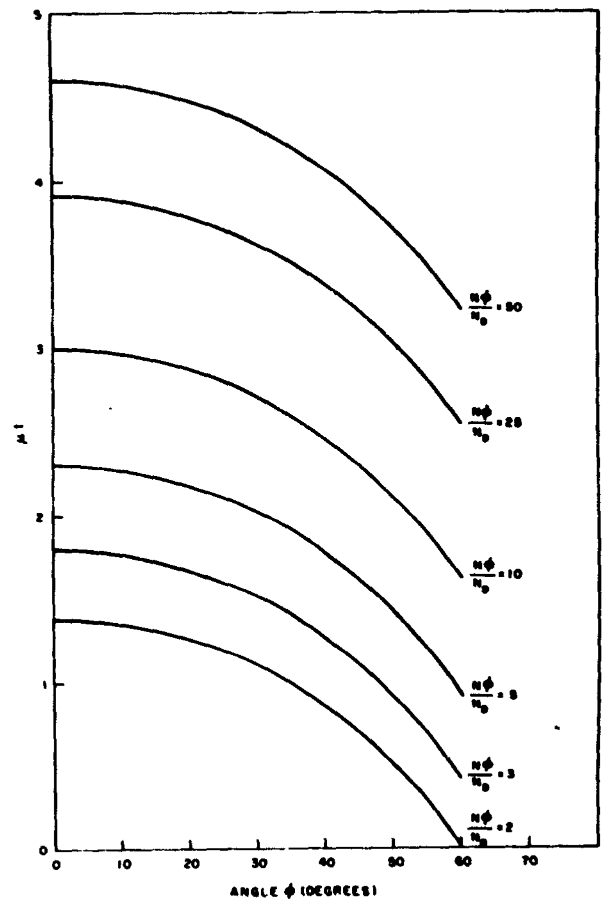

FIGURE 5 THICKNESSES OF A FLAT FACED 60° SHADOW SHIELD

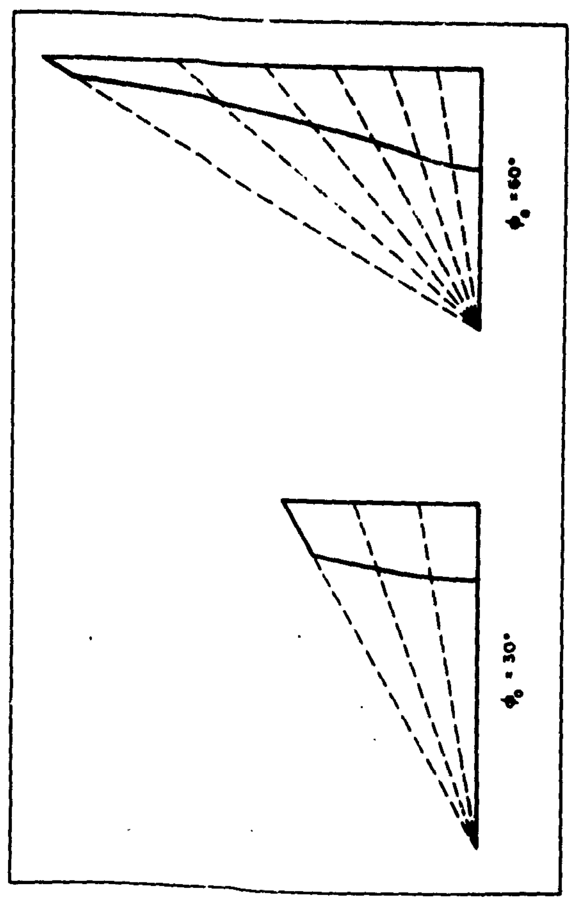

FIGURE 6. ATTENUATION OF 5

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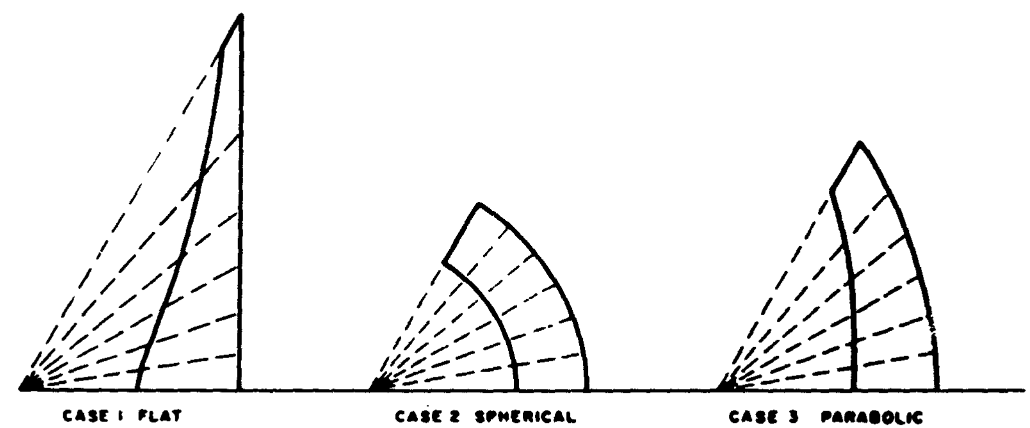

FIGURE 7. ATTENUATION OF S, $\phi_0 = 60°$

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